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MODELING OF THE 6DOF MISSILE DYNAMICS USING THE NED AXES SYSTEM

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MODELING OF THE 6DOF MISSILE DYNAMICS USING THE NED AXES SYSTEM

by

M. Lauzon

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Approved by/approuvé par

Head, Delivery Systems Section
Chef, Section systèmes de lancement

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ABSTRACT

The Defence Research Establishment Valcartier currently supports the development of a generic six-degree-of-freedom simulation model of a tactical air-to-air missile. For efficiency, the development of this missile simulation takes advantage of already existing model components when available, one example being the modeling of the missile dynamics. The documented six-degree-of-freedom missile dynamics model currently available uses a non-standard system of axes, where the x axis is forward, the z axis is lateral and the y axis is downward, to describe the missile body, seeker and missile-target line-of-sight vector. However, the most common axes system in current simulation models and in the DREV library of weapon system model components is the NED system (North-East-Down), which uses a forward x axis, a downward z axis and a lateral y axis. In order to maintain compatibility with the existing library of components, this missile dynamics model has been modified to use the NED standard system. The mathematical relations of the resulting model are presented.

RÉSUMÉ

Le Centre de recherches pour la défense de Valcartier soutient présentement le développement d'un modèle générique à six degrés de liberté d'un missile tactique air-air. Dans un but d'éfficacité, le développement de ce modèle de missile prend avantage de composantes déjà existantes, comme par exemple, dans le cas de la dynamique du missile. Le modèle de référence à six degrés de liberté de la dynamique du missile tactique utilise présentement un système d'axes différent du standard, soit l'axe x vers l'avant, l'axe y vers le bas et l'axe z en latéral, pour représenter le corps du missile, l'autodirecteur ainsi que la ligne missile-but. Le système d'axes le plus couramment utilisé dans le domaine de la simulation de missiles tactiques ainsi que dans la librairie de composantes de modèle de simulation du CRDV est le système d'axes NED, qui utilise l'axe x vers l'avant, l'axe z vers le bas et l'axe y en latéral. Pour assurer une complète compatibilité entre le modèle développé et la librairie de composantes existante, celui-ci a été modifié pour utiliser le système d'axes standard NED. Le développement des relations mathématiques du modèle modifié est présenté.

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EXECUTIVE SUMMARY

A three-degree-of-freedom computer simulation model is currently used within DND to assess the performance of tactical air-to-air missiles. This simulation capability is not of a sufficiently high fidelity to support CF-18 missiles operational capability, and a requirement for a high-fidelity simulation model has been defined. In response to this requirement, high fidelity and validated six-degree-of-freedom missile simulation models are being developed to accurately model the aerodynamics, guidance and control characteristics of current tactical air-to-air missiles for system performance assessment.

For efficiency, the development of this missile simulation takes advantage of already available model components, one example being the modeling of the missile dynamics. However, the documented six-degree-of-freedom model of the missile dynamics currently available is not based on the standard NED (North-East-Down) axes system used in most current simulation models and, in particular, in the DREV library of weapon system model components. In order to maintain compatibility with the existing library of components, this missile dynamics model has been modified to use the NED system. The definition of the missile dynamics model in the NED system represents a preliminary milestone towards the acquisition of the required simulation capability.

NOMENCLATURE

a_{xb}, a_{yb}, a_{zb}	-	missile body acceleration components (BF)
BF	-	missile body-fixed coordinate system
d_{la}	-	distance from the center of gravity to the body accelerometer
d_{miss}	-	miss distance
DND	-	Department of National Defence, Canada
DOF	-	degree-of-freedom
DREV	-	Defence Research Establishment Valcartier
EF	-	earth-fixed coordinate system
F_x, F_y, F_z	-	sum of external forces acting on the missile body
g	-	gravitational constant
h_T	-	target altitude
$H(s)$	-	missile seeker tracking loop transfer function
I_{xx}, I_{yy}, I_{zz}	-	moments of inertia of the missile body
K_p	-	tracking loop gain
LF	-	LOS-fixed coordinate system
LOS	-	missile-to-target line-of-sight
M_M	-	missile mass
M_x, M_y, M_z	-	sum of aerodynamic moments acting on the missile body
NED	-	NED system of axes
$\vec{R}$	-	Range vector
$R_{\theta_A}[\,], R_{\theta_E}[\,], R_{\theta_R}[\,]$	-	Euler rotation matrix
SF	-	missile seeker-fixed coordinate system
T_1, T_2	-	transformation matrices
TF	-	target-fixed coordinate system
u, v, w	-	components of the missile velocity, $\vec{V}_M$ (BF)
$\vec{V}_M$	-	missile velocity
$\vec{V}_T$	-	target velocity
VF	-	missile velocity-fixed coordinate system
x_i, y_i, z_i	-	cartesian axes of coordinate system i
$\hat{x}_i, \hat{y}_i, \hat{z}_i$	-	unit vectors along the cartesian axes of coordinate system i
$\hat{x}_l\|_{\hat{x}_b}$	-	contribution of $\hat{x}_b$ to $\hat{x}_l$
$\hat{x}_p$	-	unit vector perpendicular to the plane formed by $\hat{x}_s$ and $\hat{x}_l$

Subscripts

A	-	azimuth
b	-	missile body-fixed coordinate system
E	-	elevation
i1	-	intermediate I coordinate system

i2	-	intermediate II coordinate system
l	-	LOS-fixed coordinate system
M	-	missile
MT	-	missile-target (EF)
MT-l	-	missile-target (LF)
r, R	-	roll
RL	-	lateral (right-left)
s	-	missile seeker-fixed coordinate system
T	-	target
t	-	target body-fixed coordinate system
UD	-	vertical (up-down)
v	-	missile velocity-fixed coordinate system

Greek Symbols

α_A	-	projection of $\alpha_{A'}$ on the earth-fixed xy plane
$\alpha_{A'}, \alpha_E$	-	Euler angles of the missile body intermediate II coordinate system (VF)
α_{RL}, α_{UD}	-	Euler angles of the missile body (VF)
$\vec{\epsilon}$	-	total angular error vector from the seeker boresight to the LOS vector
$\epsilon_{RL}, \epsilon_{UD}$	-	Euler angles of the LOS vector (SF)
$\dot{\vec{\phi}}$	-	seeker tracking rate vector
ϕ_A, ϕ_E, ϕ_R	-	Euler angles of the missile seeker (EF)
$\dot{\vec{\phi}}_l$	-	limited seeker tracking rate vector
$\phi_r, \phi_{UD}, \phi_{RL}$	-	missile seeker angles (SF)
ϕ_{RL}, ϕ_{UD}	-	components of $\dot{\vec{\phi}}$ (SF)
$\gamma_A, \gamma_E, \gamma_R$	-	Euler angles of the missile velocity vector (EF)
$\Gamma_A, \Gamma_E, \Gamma_R$	-	Euler angles of the target velocity vector (EF)
$\lambda_{RL}, \lambda_{UD}$	-	Euler angles of the seeker (BF)
$\theta_A, \theta_E, \theta_R$	-	Euler angles of the missile body (EF)
$\theta_r, \theta_{UD}, \theta_{RL}$	-	missile body angles (BF)
$\sigma_A, \sigma_E, \sigma_R$	-	Euler angles of the LOS (EF)
$\sigma_r, \sigma_{UD}, \sigma_{RL}$	-	LOS angles (LF)
$\vec{\omega}_b$	-	missile body angular rate vector
$\vec{\omega}_l$	-	LOS angular rate vector
$\vec{\omega}_{l/s}$	-	LOS angular rate vector relative to the seeker
$\vec{\omega}_s$	-	missile seeker angular rate vector
$\vec{\omega}_{s/b}$	-	missile seeker angular rate vector relative to the missile body
$\omega_{xe}, \omega_{ye}, \omega_{ze}$	-	components of the LOS angular rate vector (EF)

1.0 INTRODUCTION

To effectively contribute to improving the CF-18 missiles operational capability, the existing simulation model and associated modeling environment in use within DND to evaluate air-to-air missile system performance require improvement with regard to fidelity, accuracy and reliability. The field of operational research and analysis would also benefit from improved models in the area of performance trade-off studies (options analysis) to support the acquisition of new missile systems. In response to this deficiency, a high fidelity and validated six-degree-of-freedom missile simulation model is currently under development to accurately model the aerodynamics, guidance and control characteristics of current tactical air-to-air missile systems.

For efficiency, the development of this missile simulation takes advantage of already existing model components when available, one example being the modeling of the missile dynamics. The available documented model of the missile six-degree-of-freedom dynamics describes in detail the motion of the body and seeker based on the missile-target relative kinematics. However, the documented model is not based on the standard NED (North-East-Down) axes system used in most current simulation models (Ref. [1]) and, in particular, in the DREV library of weapon system model components. In order to maintain compatibility with the existing library of components, the documented model of the missile dynamics has been modified to adopt the NED standard axes system. The original and NED axes systems are summarized in Table I. The Euler transformation is described in Section 2.0. The modified model of the missile dynamics is presented in detail in the following sections.

TABLE I

Definition of axes systems

Source	Axes definition	Euler transformations
Original	x forward, y up, z lateral	$\hat{y}, \hat{z}_{i1}, \hat{x}_{i2}$
NED	x forward, y lateral, z down	$\hat{z}, \hat{y}_{i1}, \hat{x}_{i2}$

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2.0 DEFINITION OF COORDINATE SYSTEMS

2.1 Euler transformations

In the standard axes system, an Euler transformation consists of three consecutive Euler rotations around the z, y and x axes respectively. Each successive rotation takes place relative to the intermediate coordinate system resulting from the previous rotation. The three rotations are expressed as three matrix operators, $R_{\theta_A}[\ ], R_{\theta_E}[\ ]$ and $R_{\theta_R}[\ ]$, which are applied in sequence. The rotation operator for θ_A represents a rotation about the z axis, and is defined as

$$R_{\theta_A}[x] = \begin{bmatrix} \cos(x) & \sin(x) & 0 \\ -\sin(x) & \cos(x) & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad [1]$$

The rotation matrix for θ_E represents a rotation about the y_{i1} axis, and is defined as

$$R_{\theta_E}[x] = \begin{bmatrix} \cos(x) & 0 & -\sin(x) \\ 0 & 1 & 0 \\ \sin(x) & 0 & \cos(x) \end{bmatrix} \quad [2]$$

and the rotation matrix for θ_R represents a rotation about the x_{i2} axis, and is defined as

$$R_{\theta_R}[x] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(x) & \sin(x) \\ 0 & -\sin(x) & \cos(x) \end{bmatrix} \quad [3]$$

The global transformation operator from the earth-fixed coordinate system (x, y, z) to any coordinate system i (x_i, y_i, z_i) is the product of these three rotation operators, namely,

$$\begin{bmatrix} \hat{x}_i \\ \hat{y}_i \\ \hat{z}_i \end{bmatrix} = R_{\theta_R}[\theta_3] \cdot R_{\theta_E}[\theta_2] \cdot R_{\theta_A}[\theta_1] \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} \quad [4]$$

where θ_1, θ_2 and θ_3 denote the three successive Euler rotations. The inverse transformation, from coordinate system i back to the earth-fixed system, is

$$\begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} = R_{\theta_A}[-\theta_1] \cdot R_{\theta_E}[-\theta_2] \cdot R_{\theta_R}[-\theta_3] \begin{bmatrix} \hat{x}_i \\ \hat{y}_i \\ \hat{z}_i \end{bmatrix} \quad [5]$$

The inverse rotation matrix operators are denoted

$$\begin{aligned} R_{\theta_A}[-\theta_1] &=> R_{-\theta_A}[\theta_1] \\ R_{\theta_E}[-\theta_2] &=> R_{-\theta_E}[\theta_2] \\ R_{\theta_R}[-\theta_3] &=> R_{-\theta_R}[\theta_3] \end{aligned} \quad [6]$$

Using the missile body Euler angles referenced to the earth-fixed coordinate system (namely, $\theta_A, \theta_E, \theta_R$) as an example, the transformation from the earth-fixed to the missile body-fixed system consists of a rotation $\theta_A \hat{z}$ followed by $\theta_E \hat{y}_{i1}$ and finally $\theta_R \hat{x}_{i2}$. The coordinate system (x, y, z) represents the initial coordinate system for the transformation, (x_{i1}, y_{i1}, z_{i1}) and (x_{i2}, y_{i2}, z_{i2}) denote coordinate systems resulting from intermediate Euler rotations, θ_A and θ_E respectively, and (x_b, y_b, z_b) denotes the final coordinate system resulting from the last rotation, θ_R. Since this last rotation is around the x_{i2} axis, $\hat{x}_{i2}$ corresponds to $\hat{x}_b$. From eq. 4, the global transformation in this example is summarized as

$$\begin{bmatrix} \hat{x}_b \\ \hat{y}_b \\ \hat{z}_b \end{bmatrix} = R_{\theta_R}[\theta_R] \cdot R_{\theta_E}[\theta_E] \cdot R_{\theta_A}[\theta_A] \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} \qquad [7]$$

2.2 Coordinate systems

The modeling of the engagement between a missile and its target consists in describing the missile and target dynamics in their respective coordinate systems and their relative kinematics in the inertial system. A minimum of six different coordinate systems are required, and they are summarized in Table II. The earth-fixed coordinate system represents

TABLE II

Primary coordinate systems

Type	Acronym	Axes
Earth-fixed	EF	(x, y, z)
Missile body-fixed	BF	(x_b, y_b, z_b)
Missile seeker-fixed	SF	(x_s, y_s, z_s)
Missile velocity vector-fixed	VF	(x_v, y_v, z_v)
Missile-target line-of-sight fixed	LF	(x_l, y_l, z_l)
Target body-fixed	TF	(x_t, y_t, z_t)

the inertial reference frame. The missile body-fixed coordinate system is attached to the missile body, and its x axis is aligned with the missile longitudinal axis, the y axis is lateral and the z axis is downwards. The missile seeker-fixed coordinate system is positioned at the seeker gyro axis of rotation. When the seeker is in the null position (boresight along the missile body), the axes of the seeker-fixed system are aligned parallel to those of the missile body-fixed system. An additional coordinate system fixed to the missile velocity vector is required to compute the components of the missile incidence. Also, a coordinate system is attached to the missile-to-target LOS vector, with its x axis along the LOS vector, to compute the missile-target relative kinematics. Finally, a coordinate system is attached to the target to model its dynamics and kinematics.

Euler transformations result in two intermediate coordinate systems. The notation for these two systems is summarized in Table III. For simplicity, this notation is used in the present document to identify the intermediate coordinate systems of all Euler transformations. Accordingly, the systems (x_{i1}, y_{i1}, z_{i1}) and (x_{i2}, y_{i2}, z_{i2}) refer to different coordinate systems for different Euler transformations.

TABLE III

Intermediate coordinate systems

Type	Notation	Axes
Intermediate I	I1	(x_{i1}, y_{i1}, z_{i1})
Intermediate II	I2	(x_{i2}, y_{i2}, z_{i2})

The Euler transformations between the various coordinate systems of Table II are summarized in Tables IV, V, VI and VII, where the Euler rotations are shown in order. Table IV presents the Euler transformations from the earth-fixed coordinate system to each of the other five systems, while Table V summarizes the transformations from the missile body to the seeker and missile velocity vector. Table VI presents the transformation from the missile body intermediate II coordinate system to the missile velocity vector. The missile body intermediate II system is obtained by performing only the first two rotations of the inertial-to-missile body transformation (eq. 7), namely, rotations θ_A and θ_E in sequence. Finally, Table VII presents the transformation from the missile seeker to the missile-target LOS, which is used in the computation of the seeker tracking error.

TABLE IV

Euler transformations referenced to the inertial system

Name	Rotations
missile body	$\theta_A\hat{z}, \theta_E\hat{y}_{i1}, \theta_R\hat{x}_{i2}$ ($\hat{x}_{i2}$ is $\hat{x}_b$)
seeker	$\phi_A\hat{z}, \phi_E\hat{y}_{i1}, \phi_R\hat{x}_{i2}$ ($\hat{x}_{i2}$ is $\hat{x}_s$)
missile velocity vector	$\gamma_A\hat{z}, \gamma_E\hat{y}_{i1}, \gamma_R\hat{x}_{i2}$ ($\hat{x}_{i2}$ is $\hat{x}_v$)
LOS	$\sigma_A\hat{z}, \sigma_E\hat{y}_{i1}, \sigma_R\hat{x}_{i2}$ ($\hat{x}_{i2}$ is $\hat{x}_l$)
target velocity vector	$\Gamma_A\hat{z}, \Gamma_E\hat{y}_{i1}, \Gamma_R\hat{x}_{i2}$ ($\hat{x}_{i2}$ is $\hat{x}_t$)

TABLE V

Euler transformations referenced to the missile body-fixed system

Name	Rotations
missile seeker	$\epsilon_{RL}\hat{z}_b, \epsilon_{UD}\hat{y}_{i1}$, $0\hat{x}_{i2}(\hat{x}_{i2}$ is $\hat{x}_s)$
missile velocity vector	$-\alpha_{RL}\hat{z}_b, -\alpha_{UD}\hat{y}_{i1}$, $0\hat{x}_{i2}$ $(\hat{x}_{i2}$ is $\hat{x}_v)$

TABLE VI

Euler transformations referenced to the missile body intermediate II system

Name	Rotations
missile velocity vector	$-\alpha_{A'}\hat{z}_b, -\alpha_E\hat{y}_{i1}$, $0\hat{x}_{i2}$ $(\hat{x}_{i2}$ is $\hat{x}_v)$

Note: The intermediate II system is obtained from the transformation $R_{\theta_E}[\theta_E].R_{\theta_A}[\theta_A].[\hat{x}, \hat{y}, \hat{z}]'$

TABLE VII

Euler transformations referenced to the missile seeker system

Name	Rotations
LOS	$\lambda_{RL}\hat{z}_s, \lambda_{UD}\hat{y}_{i1}$, $0\ \hat{x}_{i2}$ $(\hat{x}_{i2}$ is $\hat{x}_l)$

The Euler transformations of Tables IV, V, VI and VII have the following particularities:

- No angular $\hat{x}_{i2}$ ($\hat{x}_s$) rotation from the body-fixed to the seeker-fixed systems (Table V)
- No angular $\hat{x}_{i2}$ ($\hat{x}_v$) rotation from the body-fixed to the missile velocity-fixed systems (Table V)
- No angular $\hat{x}_{i2}$ ($\hat{x}_v$) rotation from the body-fixed intermediate II to the missile velocity-fixed systems (Table VI)
- No angular $\hat{x}_{i2}$ ($\hat{x}_l$) rotation from the seeker-fixed to the LOS-fixed systems (Table VII)

The non-Eulerian angular rates of the missile body, missile seeker and missile-target LOS are defined in Table VIII. These rates are not Euler angular rates because the rotations take place around the three axes of the same coordinate system, namely, the body-fixed, seeker-fixed and LOS-fixed system respectively.

TABLE VIII

Non-Eulerian angular rates

Name	Angular rates
missile body	$\dot{\theta}_r \hat{x}_b, \dot{\theta}_{UD} \hat{y}_b, \dot{\theta}_{RL} \hat{z}_b$
missile seeker	$\dot{\phi}_r \hat{x}_s, \dot{\phi}_{UD} \hat{y}_s, \dot{\phi}_{RL} \hat{z}_s$
LOS	$\dot{\sigma}_r \hat{x}_l, \dot{\sigma}_{UD} \hat{y}_l, \dot{\sigma}_{RL} \hat{z}_l$

The mathematical relations of the six-degree-of-freedom missile dynamics model are next developed using the standard NED axes system (Ref. [1] and Table I) . The discussion also includes two issues of importance in missile-target engagement modeling, namely, the computation of the initial conditions and of the miss distance.

3.0 MISSILE DYNAMICS

3.1 Body coordinate system

When referenced to the inertial coordinate system, the missile body angular rate vector, denoted $\vec{\omega}_b$, can be defined in terms of the Eulerian angular rotations $\dot{\theta}_A$, $\dot{\theta}_E$ and $\dot{\theta}_R$, namely,

$$\vec{\omega}_b = \dot{\theta}_R\hat{x}_{i2} + \dot{\theta}_E\hat{y}_{i1} + \dot{\theta}_A\hat{z} \qquad [8]$$

The inertial and missile body-fixed coordinate systems are related by the Euler transformation based on the angles $\theta_A, \theta_E, \theta_R$ (eq. 7). Equation 8 can therefore be converted to the body-fixed system using this same transformation. The transformation from the (x, y, z) system to the body-fixed one requires three rotations,. Two rotations are required from (x_{i1}, y_{i1}, z_{i1}) to (x_b, y_b, z_b) and one rotation from (x_{i2}, y_{i2}, z_{i2}). Accordingly, the $\hat{z}$ component of $\vec{\omega}_b$ in eq. 8, $\dot{\theta}_A$, must be rotated through all three rotations. The $\hat{y}_{i1}$ component, $\dot{\theta}_E$, must be rotated through two rotations, namely, θ_E and θ_R. The $\hat{x}_{i2}$ component, $\dot{\theta}_R$, only needs to be rotated through θ_R. In summary, the transformation of the three components of eq. 8 to the body-fixed system is

$$\vec{\omega}_b = R_{\theta_R}[\theta_R].R_{\theta_E}[\theta_E].R_{\theta_A}[\theta_A].\begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_A \end{bmatrix} + R_{\theta_R}[\theta_R].R_{\theta_E}[\theta_E].\begin{bmatrix} 0 \\ \dot{\theta}_E \\ 0 \end{bmatrix} + R_{\theta_R}[\theta_R].\begin{bmatrix} \dot{\theta}_R \\ 0 \\ 0 \end{bmatrix} \qquad [9]$$

Substituting for the definition of the transformation operators (eqs. 1, 2, 3) in eq. 9,

$$\begin{aligned}\vec{\omega}_b = \; & \{-\sin[\theta_E]\dot{\theta}_A + \dot{\theta}_R\}\hat{x}_b + \{\cos[\theta_E]\sin[\theta_R]\dot{\theta}_A + \cos[\theta_R]\dot{\theta}_E\}\hat{y}_b + \\ & \{\cos[\theta_E]\cos[\theta_R]\dot{\theta}_A - \sin[\theta_R]\dot{\theta}_E\}\hat{z}_b \end{aligned} \qquad [10]$$

The missile body angular rate vector $\vec{\omega}_b$ is also defined in the body-fixed coordinate system as

$$\vec{\omega}_b = \dot{\theta}_r\hat{x}_b + \dot{\theta}_{UD}\hat{y}_b + \dot{\theta}_{RL}\hat{z}_b \qquad [11]$$

Equating the $\hat{x}_b, \hat{y}_b, \hat{z}_b$ components of eqs. 10 and 11,

$$\dot{\theta}_A = \sec[\theta_E](\cos[\theta_R]\dot{\theta}_{RL} + \sin[\theta_R]\dot{\theta}_{UD}) \qquad [12]$$

$$\dot{\theta}_E = -\sin[\theta_R]\dot{\theta}_{RL} + \cos[\theta_R]\dot{\theta}_{UD} \qquad [13]$$

$$\dot{\theta}_R = \sin[\theta_E]\dot{\theta}_A + \dot{\theta}_r \qquad [14]$$

Equations 12, 13 and 14 define the missile body Euler angular rates $(\dot{\theta}_A, \dot{\theta}_E, \dot{\theta}_R)$ referenced to the earth-fixed coordinate system in terms of the non-Eulerian rates $(\dot{\theta}_r, \dot{\theta}_{UD}, \dot{\theta}_{RL})$ referenced to the body-fixed system. The body-fixed rates $(\dot{\theta}_r, \dot{\theta}_{UD}, \dot{\theta}_{RL})$ are computed from the traditional missile equations of motion (Section 3.5).

3.2 Seeker coordinate system

The seeker angular rate vector, denoted $\vec{\omega}_s$, is expressed in terms of the Euler angular rotations $(\dot{\phi}_A, \dot{\phi}_E, \dot{\phi}_R)$ in reference to the earth-fixed system as

$$\vec{\omega}_s = \dot{\phi}_R \hat{x}_{i2} + \dot{\phi}_E \hat{y}_{i1} + \dot{\phi}_A \hat{z} \quad [15]$$

$\vec{\omega}_s$ is also expressed in the seeker-fixed coordinate system as

$$\vec{\omega}_s = \dot{\phi}_r \hat{x}_s + \dot{\phi}_{UD} \hat{y}_s + \dot{\phi}_{RL} \hat{z}_s \quad [16]$$

The conversion of eq. 15 to the seeker-fixed system is obtained using a procedure identical to the one presented in Section 3.1 for $\vec{\omega}_b$, except that the Euler angles are now ϕ_A, ϕ_E and ϕ_R:

$$\vec{\omega}_s = R_{\theta_R}[\phi_R].R_{\theta_E}[\phi_E].R_{\theta_A}[\phi_A]. \begin{bmatrix} 0 \\ 0 \\ \dot{\phi}_A \end{bmatrix} + R_{\theta_R}[\phi_R].R_{\theta_E}[\phi_E]. \begin{bmatrix} 0 \\ \dot{\phi}_E \\ 0 \end{bmatrix} + R_{\theta_R}[\phi_R]. \begin{bmatrix} \dot{\phi}_R \\ 0 \\ 0 \end{bmatrix} \quad [17]$$

or,

$$\begin{aligned} \vec{\omega}_s \;=\; & \{\dot{\phi}_R - \sin[\phi_E]\dot{\phi}_A\}\hat{x}_s + \{\cos[\phi_E]\sin[\phi_R]\dot{\phi}_A + \cos[\phi_R]\dot{\phi}_E\}\hat{y}_s \\ & + \{\cos[\phi_E]\cos[\phi_R]\dot{\phi}_A - \sin[\phi_R]\dot{\phi}_E\}\hat{z}_s \end{aligned} \quad [18]$$

Equating the $\hat{x}_s, \hat{y}_s, \hat{z}_s$ components of eqs. 16 and 18,

$$\dot{\phi}_A = \sec[\phi_E](\cos[\phi_R]\dot{\phi}_{RL} + \sin[\phi_R]\dot{\phi}_{UD}) \quad [19]$$

$$\dot{\phi}_E = -\sin[\phi_R]\dot{\phi}_{RL} + \cos[\phi_R]\dot{\phi}_{UD} \quad [20]$$

$$\dot{\phi}_R = \sin[\phi_E]\dot{\phi}_A + \dot{\phi}_r \quad [21]$$

Equations 19, 20 and 21 define the Euler angular rates $(\dot{\phi}_A, \dot{\phi}_E, \dot{\phi}_R)$ referenced to the earth-fixed system in terms of the non-Eulerian rates $(\dot{\phi}_r, \dot{\phi}_{UD}, \dot{\phi}_{RL})$ referenced to the seeker-fixed system. The angles $\dot{\phi}_{UD}$ and $\dot{\phi}_{RL}$ are computed as part of the tracking loop (Section 4.0), while $\dot{\phi}_r$, the seeker spin rate, is obtained in Section 3.3.

3.3 Seeker-to-body coordinate transformations

This Euler transformation is more complex than those previously discussed because the initial coordinate system for the transformation, namely, the body-fixed coordinate system, is a rotating one while, previously, the earth-fixed system was stationary. The body angular rate $\vec{\omega}_b$ is related to the seeker angular rate $\vec{\omega}_s$ according to

$$\vec{\omega}_s = \vec{\omega}_b + \vec{\omega}_{s/b} \quad [22]$$

where the angular rate of the seeker relative to the body is denoted as $\overline{\omega}_{s/b}$. The terms of eq. 22 are

$$\vec{\omega}_b = \dot{\theta}_r \hat{x}_b + \dot{\theta}_{UD} \hat{y}_b + \dot{\theta}_{RL} \hat{z}_b \qquad [23]$$

$$\vec{\omega}_s = \dot{\phi}_r \hat{x}_s + \dot{\phi}_{UD} \hat{y}_s + \dot{\phi}_{RL} \hat{z}_s \qquad [24]$$

and

$$\vec{\omega}_{s/b} = \dot{\lambda}_{UD} \hat{y}_{i1} + \dot{\lambda}_{RL} \hat{z}_b \qquad [25]$$

It is noted that there are no $\hat{x}_{i2}$ angular rotation from the body-fixed coordinate system to the seeker-fixed system. The rotations are $\lambda_{RL}\hat{z}_b$ followed by $\lambda_{UD}\hat{y}_{i1}$ in sequence. Equations 23, 24 and 25 are converted to a single coordinate system, namely, the seeker-fixed system. The terms in eq. 23 refer to the body-fixed system and are therefore converted to the seeker-fixed one according to

$$\vec{\omega}_b = R_{\theta_E}[\lambda_{UD}].R_{\theta_A}[\lambda_{RL}].\begin{bmatrix} \dot{\theta}_r \\ \dot{\theta}_{UD} \\ \dot{\theta}_{RL} \end{bmatrix} \qquad [26]$$

or

$$\begin{aligned} \vec{\omega}_b = \; & \{\cos[\lambda_{RL}]\cos[\lambda_{UD}]\dot{\theta}_r - \sin[\lambda_{UD}]\dot{\theta}_{RL} + \cos[\lambda_{UD}]\sin[\lambda_{RL}]\dot{\theta}_{UD}\}\hat{x}_s + \\ & \{-\sin[\lambda_{RL}]\dot{\theta}_r + \cos[\lambda_{RL}]\dot{\theta}_{UD}\}\hat{y}_s + \\ & \{\cos[\lambda_{RL}]\sin[\lambda_{UD}]\dot{\theta}_r + \cos[\lambda_{UD}]\dot{\theta}_{RL} + \sin[\lambda_{RL}]\sin[\lambda_{UD}]\dot{\theta}_{UD}\}\hat{z}_s \end{aligned} \qquad [27]$$

The term $\vec{\omega}_s$ requires no transformation. The term $\vec{\omega}_{s/b}$ is transformed to the seeker-fixed coordinate system using

$$\vec{\omega}_{s/b} = R_{\theta_E}[\lambda_{UD}].R_{\theta_A}[\lambda_{RL}].\begin{bmatrix} 0 \\ 0 \\ \dot{\lambda}_{RL} \end{bmatrix} + R_{\theta_E}[\lambda_{UD}].\begin{bmatrix} 0 \\ \dot{\lambda}_{UD} \\ 0 \end{bmatrix} \qquad [28]$$

or

$$\vec{\omega}_{s/b} = -\sin[\lambda_{UD}]\dot{\lambda}_{RL}\hat{x}_s + \dot{\lambda}_{UD}\hat{y}_s + \cos[\lambda_{UD}]\dot{\lambda}_{RL}\hat{z}_s \qquad [29]$$

Substituting eqs. 24, 27 and 29 in eq. 22, and equating the three components $\hat{x}_s, \hat{y}_s$ and $\hat{z}_s$,

$$\dot{\lambda}_{UD} = \sin[\lambda_{RL}]\dot{\theta}_r - \cos[\lambda_{RL}]\dot{\theta}_{UD} + \dot{\phi}_{UD} \qquad [30]$$

$$\begin{aligned} \dot{\lambda}_{RL} = \; & -\sec[\lambda_{UD}](\cos[\lambda_{RL}]\sin[\lambda_{UD}]\dot{\theta}_r + \cos[\lambda_{UD}]\dot{\theta}_{RL} + \\ & \sin[\lambda_{RL}]\sin[\lambda_{UD}]\dot{\theta}_{UD} - \dot{\phi}_{RL}) \end{aligned} \qquad [31]$$

$$\dot{\phi}_r = \cos[\lambda_{RL}]\cos[\lambda_{UD}]\dot{\theta}_r - \sin[\lambda_{UD}]\dot{\theta}_{RL} + \cos[\lambda_{UD}]\sin[\lambda_{RL}]\dot{\theta}_{UD} - \sin[\lambda_{UD}]\dot{\lambda}_{RL} \qquad [32]$$

$\dot{\lambda}_{UD}, \dot{\lambda}_{RL}$ and $\dot{\phi}_r$ describe the seeker dynamics relative to the missile body.

3.4 Velocity-fixed coordinate system

The x axis of the coordinate system fixed to the missile velocity vector $\vec{V}_M$ is aligned with the velocity vector. Also, the missile velocity vector is described by the Euler angles γ_A, γ_E and γ_R referenced to the earth-fixed system. Accordingly, the velocity vector in the velocity-fixed system is converted to the earth-fixed system using the inverse Euler rotations $-\gamma_R$, $-\gamma_E$ and $-\gamma_A$ in sequence. Using the inverse transformation operators described in Section 2.1,

$$\vec{V}_M = R_{-\theta_A}[\gamma_A].R_{-\theta_E}[\gamma_E].R_{-\theta_R}[\gamma_R].\begin{bmatrix} V_M \\ 0 \\ 0 \end{bmatrix} \qquad [33]$$

or

$$\vec{V}_M = \cos[\gamma_A]\cos[\gamma_E]V_M\hat{x} + \cos[\gamma_E]\sin[\gamma_A]V_M\hat{y} - \sin[\gamma_E]V_M\hat{z} \qquad [34]$$

The missile body Euler angles referenced to the earth-fixed system are $(\theta_A, \theta_E, \theta_R)$. Rotations through θ_A and θ_E in sequence result in the intermediate reference system (x_{i2}, y_{i2}, z_{i2}), where x_{i2} corresponds to the missile body longitudinal axis, x_b. The angle between the missile velocity vector $\vec{V}_M$ and the body projected on the x_{i2}, y_{i2} plane is defined as $\alpha_{A'}$. The angle perpendicular to the x_{i2}, y_{i2} plane is defined as α_E. α_A is defined as the projection of $\alpha_{A'}$ onto the earth-fixed x, y plane. A final rotation θ_R about x_{i2} results in the body coordinate system (x_b, y_b, z_b). α_{RL} is defined as the angle between the velocity vector and the missile body in the x_b, y_b plane, and α_{UD} is the angle perpendicular to the x_b, y_b plane.

Using the above definitions, the missile velocity vector can be expressed in both the (x_b, y_b, z_b) and (x_{i2}, y_{i2}, z_{i2}) coordinate systems. Referenced to the body, the velocity vector is

$$\vec{V}_M = R_{-\theta_A}[-\alpha_{RL}].R_{-\theta_E}[-\alpha_{UD}].\begin{bmatrix} V_M \\ 0 \\ 0 \end{bmatrix} \qquad [35]$$

The transformation to (x_{i2}, y_{i2}, z_{i2}) from the earth-fixed system is performed using the operator $R_{\theta_E}[\theta_E].R_{\theta_A}[\theta_A]$, and the last rotation, $R_{\theta_R}[\theta_R]$, converts to (x_b, y_b, z_b). Accordingly, the complete transformation of the velocity vector through (x_b, y_b, z_b) to (x_{i2}, y_{i2}, z_{i2}) is

$$\vec{V}_M = R_{-\theta_R}[\theta_R].R_{-\theta_A}[-\alpha_{RL}].R_{-\theta_E}[-\alpha_{UD}].\begin{bmatrix} V_M \\ 0 \\ 0 \end{bmatrix} \qquad [36]$$

or

$$\vec{V}_M = \cos[\alpha_{RL}]\cos[\alpha_{UD}]V_M\hat{x}_{i2} + (-\cos[\alpha_{UD}]\cos[\theta_R]\sin[\alpha_{RL}] - \sin[\alpha_{UD}]\sin[\theta_R])V_M\hat{y}_{i2} + (\cos[\theta_R]\sin[\alpha_{UD}] - \cos[\alpha_{UD}]\sin[\alpha_{RL}]\sin[\theta_R])V_M\hat{z}_{i2} \quad [37]$$

It is noted that the negative of the incidence angles α_{RL} and α_{UD} are used in the above transformation since the incidence is measured from the vector $\vec{V}_M$ to the missile body. The inverse rotation, from the body to $\vec{V}_M$, therefore uses negative α's. Similarly, the velocity vector referenced to the system (x_{i2}, y_{i2}, z_{i2}) is

$$\vec{V}_M = R_{-\theta_A}[-\alpha_{A'}].R_{-\theta_E}[-\alpha_E].\begin{bmatrix} V_M \\ 0 \\ 0 \end{bmatrix} \quad [38]$$

or

$$\vec{V}_M = \cos[\alpha_E]\cos[\alpha_{A'}]V_M\hat{x}_{i2} - \cos[\alpha_E]\sin[\alpha_{A'}]V_M\hat{y}_{i2} + \sin[\alpha_E]V_M\hat{z}_{i2} \quad [39]$$

The angles $\alpha_{A'}$ and α_E are solved from eqs. 37 and 39,

$$\sin[\alpha_E] = \cos[\theta_R]\sin[\alpha_{UD}] - \cos[\alpha_{UD}]\sin[\alpha_{RL}]\sin[\theta_R] \quad [40]$$

and

$$\sin[\alpha_{A'}] = \sec[\alpha_E](\cos[\alpha_{UD}]\cos[\theta_R]\sin[\alpha_{RL}] + \sin[\alpha_{UD}]\sin[\theta_R]) \quad [41]$$

From geometrical considerations,

$$\sin[\alpha_{A'}] = \sin[\alpha_A]\cos[\gamma_E] \quad [42]$$

If the missile incidence components α_{UD}, α_{RL}, α_E and $\alpha_{A'}$ remain small (less than 10 degrees), eqs. 40 and 41 can be simplified, after incorporating eq. 42, to

$$\alpha_E = -\sin[\theta_R]\alpha_{RL} + \cos[\theta_R]\alpha_{UD} \quad [43]$$

and

$$\alpha_A = \sec[\gamma_E](\cos[\theta_R]\alpha_{RL} + \sin[\theta_R]\alpha_{UD}) \quad [44]$$

where γ_A, γ_E are the velocity vector Euler angles relative to the earth-fixed system,

$$\gamma_A = \theta_A - \alpha_A \quad [45]$$

$$\gamma_E = \theta_E - \alpha_E \quad [46]$$

3.5 Equations of motion

3.5.1 Translational equations

The equations of motions in translation are

$$F_x = M_M(\dot{u} - \dot{\theta}_{RL}v + \dot{\theta}_{UD}w) \quad [47]$$

$$F_y = M_M(\dot{v} + \dot{\theta}_{RL}u - \dot{\theta}_r w) \quad [48]$$

$$F_z = M_M(\dot{w} + \dot{\theta}_r v - \dot{\theta}_{UD}u) \quad [49]$$

Also, since

$$\vec{V}_M = u\hat{x}_b + v\hat{y}_b + w\hat{z}_b \quad [50]$$

then,

$$V_M = \left|\vec{V}_M\right| = \sqrt{u^2 + v^2 + w^2} \quad [51]$$

The acceleration is given by

$$\dot{V}_M = \frac{(u\dot{u} + v\dot{v} + w\dot{w})}{V_M} \quad [52]$$

Solving for $\dot{u}, \dot{v}, \dot{w}$ from eqs. 47, 48, 49,

$$\dot{u} = \frac{F_x + M_M v\dot{\theta}_{RL} - M_M w\dot{\theta}_{UD}}{M_M} \quad [53]$$

$$\dot{v} = \frac{F_y + M_M w\dot{\theta}_r - M_M u\dot{\theta}_{RL}}{M_M} \quad [54]$$

$$\dot{w} = \frac{F_z - M_M v\dot{\theta}_r + M_M u\dot{\theta}_{UD}}{M_M} \quad [55]$$

Substituting eqs. 53, 54 and 55 into eq. 52,

$$\dot{V}_M = \frac{F_x u + F_y v + F_z w}{M_M V_M} \quad [56]$$

Since the rotation from the body to the missile velocity vector $\vec{V}_M$ is $-\alpha_{RL}$ followed by $-\alpha_{UD}$, the velocity vector is converted to the body-fixed system using the transformation $R_{\theta_E}[-\alpha_{RL}] \cdot R_{\theta_A}[-\alpha_{UD}]$, or

$$\vec{V}_M = \cos[\alpha_{RL}]\cos[\alpha_{UD}]V_M\hat{x}_b - \cos[\alpha_{UD}]\sin[\alpha_{RL}]V_M\hat{y}_b + \sin[\alpha_{UD}]V_M\hat{z}_b \quad [57]$$

Using small angle approximations,

$$\vec{V}_M = V_M\hat{x}_b - V_M\alpha_{RL}\hat{y}_b + V_M\alpha_{UD}\hat{z}_b \quad [58]$$

The values of u, v, w are obtained by comparing eqs. 50 and 58. Substituting in eq. 56,

$$\dot{V}_M = \frac{F_x - F_y\alpha_{RL} + F_z\alpha_{UD}}{M_M} \quad [59]$$

The components of the missile incidence are

$$\alpha_{RL} = -\frac{v}{V_M} \quad [60]$$

and

$$\alpha_{UD} = \frac{w}{V_M} \quad [61]$$

Equations 60 and 61 are differentiated with respect to time to obtain expressions for $\dot{\alpha}_{RL}$ and $\dot{\alpha}_{UD}$. Substituting eq. 54 for $\dot{v}$, eq. 55 for $\dot{w}$ and eq. 56 for $\dot{V}_M$,

$$\dot{\alpha}_{RL} = \dot{\theta}_{RL} - \frac{F_y}{M_M V_M} - \frac{F_x\alpha_{RL}}{M_M V_M} - \dot{\theta}_r\alpha_{UD} \quad [62]$$

and

$$\dot{\alpha}_{UD} = \dot{\theta}_{UD} + \frac{F_z}{M_M V_M} + \dot{\theta}_r\alpha_{RL} - \frac{F_x\alpha_{UD}}{M_M V_M} \quad [63]$$

3.5.2 Rotational equations

Given that the missile is symmetric and neglecting all products of inertia terms,

$$\ddot{\theta}_r = M_x/I_{xx} \quad [64]$$

$$\ddot{\theta}_{UD} = (M_y - (I_{xx} - I_{zz})\dot{\theta}_{RL}\dot{\theta}_r)/I_{yy} \quad [65]$$

$$\ddot{\theta}_{RL} = (M_z - (I_{yy} - I_{xx})\dot{\theta}_{UD}\dot{\theta}_r)/I_{zz} \quad [66]$$

where M_x, M_y, M_z denote the aerodynamic moments and I_{xx}, I_{yy}, I_{zz}, the missile airframe moments of inertia about the three primary body axes.

3.6 Accelerometer output

The conversion of the gravity force $\vec{g}$ from the earth-fixed reference system to the body-fixed one is

$$\vec{g} = R_{\theta_R}[\theta_R].R_{\theta_E}[\theta_E].R_{\theta_A}[\theta_A].\{0, 0, g\} \quad [67]$$

or

$$\vec{g} = -g\sin[\theta_E]\hat{x}_b + g\cos[\theta_E]\sin[\theta_R]\hat{y}_b + g\cos[\theta_E]\cos[\theta_R]\hat{z}_b \qquad [68]$$

The components of the gravity force in the body-fixed system are subtracted from the net missile body force components to reflect the output of the accelerometer. The longitudinal and lateral accelerations measured by the accelerometer, in units of g's, are

$$\begin{bmatrix} a_{x_b} \\ a_{y_b} \\ a_{z_b} \end{bmatrix} = \begin{bmatrix} \frac{F_x + g\sin[\theta_E]M_M}{gM_M} \\ \frac{F_y - g\cos[\theta_E]\sin[\theta_R]M_M}{gM_M} - \frac{d_{la}\ddot{\theta}_{RL}}{g} \\ \frac{F_z - g\cos[\theta_E]\cos[\theta_R]M_M}{gM_M} + \frac{d_{la}\ddot{\theta}_{UD}}{g} \end{bmatrix} \qquad [69]$$

where d_{la} is the distance from the center of gravity to the accelerometer.

4.0 TRACKING LOOP DYNAMICS

4.1 Target kinematics

The target-velocity $\vec{V}_T$ is assumed to be aligned with its body at all times during its flight. The x axis of the target-fixed coordinate system is therefore aligned with $\vec{V}_T$. The target velocity is also described by the Euler angles Γ_A, Γ_E and Γ_R, referenced to the earth-fixed system. Accordingly, the target velocity in the target-fixed system is converted to the earth-fixed system using the inverse Euler rotations $-\Gamma_R$, $-\Gamma_E$ and $-\Gamma_A$ in sequence. Using the inverse transformation operators described in Section 2.1,

$$\vec{V}_T = R_{-\theta_A}[\Gamma_A].R_{-\theta_E}[\Gamma_E].R_{-\theta_R}[\Gamma_R].\begin{bmatrix} V_T \\ 0 \\ 0 \end{bmatrix} \qquad [70]$$

or

$$\vec{V}_T = \cos[\Gamma_A]\cos[\Gamma_E]V_T\hat{x} + \cos[\Gamma_E]\sin[\Gamma_A]V_T\hat{y} - \sin[\Gamma_E]V_T\hat{z} \qquad [71]$$

4.2 LOS Euler angles and angular rates

4.2.1 LOS Euler angles

The Euler angles of the missile-target LOS vector are computed from the missile-target relative geometry in the earth-fixed coordinate system. The relative missile-target range vector $\vec{R}_{MT}$ has components (x_{MT}, y_{MT}, z_{MT}) in the inertial system and $(x_{MT-l}, y_{MT-l}, z_{MT-l})$ in the LOS-fixed system. Since the x_l axis is aligned with the LOS vector, the y_l and z_l components of the range vector, namely y_{MT-l}, z_{MT-l}, are null and the x_l component is the missile-target range, R_{MT}. Accordingly, the range vector is, by inspection,

$$\vec{R}_{MT} = \cos[\sigma_A]\cos[\sigma_E]R_{MT}\hat{x} + \cos[\sigma_E]\sin[\sigma_A]R_{MT}\hat{y} - \sin[\sigma_E]R_{MT}\hat{z} = R_{MT}\hat{x}_l \qquad [72]$$

The Euler angles σ_A, σ_E are obtained by inspection of the geometry,

$$\sigma_A = \tan^{-1}[\frac{y_{MT}}{x_{MT}}] \qquad [73]$$

and

$$\sigma_E = -\sin^{-1}[\frac{z_{MT}}{R_{MT}}] \qquad [74]$$

The coordinates of the LOS vector in the LOS-fixed coordinate system are obtained from

$$\begin{bmatrix} x_{MT-l} \\ y_{MT-l} \\ x_{MT-l} \end{bmatrix} = R_{\theta_R}[\sigma_R].R_{\theta_E}[\sigma_E].R_{\theta_A}[\sigma_A]. \begin{bmatrix} x_{MT} \\ y_{MT} \\ z_{MT} \end{bmatrix} \qquad [75]$$

where, for instance, x_{MT} is $x_T - x_M$. Substituting eqs. 73 and 74 for σ_A and σ_E respectively in eq. 75,

$$x_{MT-l} = R_{MT}, \quad y_{MT-l} = 0, \quad z_{MT-l} = 0 \qquad [76]$$

as expected.

The expression for σ_R is obtained from the earth-fixed to LOS-fixed Euler transformation. This transformation is written as a combination of two transformations, from the earth-fixed system to the seeker-fixed one followed by one from the seeker-fixed system to the LOS-fixed one, namely,

$$T_1 = R_{\theta_E}[\epsilon_{UD}].R_{\theta_A}[\epsilon_{RL}].R_{\theta_R}[\phi_R].R_{\theta_E}[\phi_E].R_{\theta_A}[\phi_A] \qquad [77]$$

The transformation from the earth-fixed to the LOS-fixed coordinate systems can also be written in terms of the LOS Euler angles referenced to the earth-fixed system, namely,

$$T_2 = R_{\theta_R}[\sigma_R].R_{\theta_E}[\sigma_E].R_{\theta_A}[\sigma_A] \qquad [78]$$

From eq. 78, the ratio of $T_2(2,3)$ over $T_2(3,3)$ gives $\tan[\sigma_R]$ since the two terms are $\cos[\sigma_E]\sin[\sigma_R]$ and $\cos[\sigma_E]\cos[\sigma_R]$ respectively (derivation not shown). Since T_1 (eq. 77) and T_2 (eq. 78) are equivalent, the complete expression for $\tan[\sigma_R]$ is given by the ratio of $T_1(2,3)$ over $T_1(3,3)$, namely,

$$\begin{aligned} \tan[\sigma_R] = \; & (\sin[\epsilon_{RL}]\sin[\phi_E] + \cos[\epsilon_{RL}]\cos[\phi_E]\sin[\phi_R])/ \\ & (-\cos[\epsilon_{RL}]\sin[\epsilon_{UD}]\sin[\phi_E] + \cos[\phi_E] \\ & (\cos[\epsilon_{UD}]\cos[\phi_R] + \sin[\epsilon_{RL}]\sin[\epsilon_{UD}]\sin[\phi_R])) \end{aligned} \qquad [79]$$

Equation 79 can be simplified for small angles ϵ_{RL} and ϵ_{UD},

$$\tan[\sigma_R] = \frac{\cos[\phi_E]\sin[\phi_R] + \sin[\phi_E]\epsilon_{RL}}{\cos[\phi_E]\cos[\phi_R] - \sin[\phi_E]\epsilon_{UD}} \qquad [80]$$

4.2.2 LOS Eulerian angular rates

The relative missile-target velocity vector $\vec{V}_{MT}$ in the earth-fixed system is defined as

$$\vec{V}_{MT} = \dot{x}_{MT}\hat{x} + \dot{y}_{MT}\hat{y} + \dot{z}_{MT}\hat{z} \qquad [81]$$

The relative velocity vector referenced to the earth-fixed system can be converted to the LOS-fixed system using

$$\vec{V}_{MT} = R_{\theta_R}[\sigma_R].R_{\theta_E}[\sigma_E].R_{\theta_A}[\sigma_A].\begin{bmatrix} \dot{x}_{MT} \\ \dot{y}_{MT} \\ \dot{z}_{MT} \end{bmatrix} \quad [82]$$

or

$$\begin{aligned}\vec{V}_{MT} \;=\; & \{\cos[\sigma_A]\cos[\sigma_E]\dot{x}_{MT} + \cos[\sigma_E]\sin[\sigma_A]\dot{y}_{MT} - \sin[\sigma_E]\dot{z}_{MT}\}\hat{x}_l + \\ & \{(-\cos[\sigma_R]\sin[\sigma_A] + \cos[\sigma_A]\sin[\sigma_E]\sin[\sigma_R])\dot{x}_{MT} + (\cos[\sigma_A]\cos[\sigma_R] + \sin[\sigma_A]\sin[\sigma_E] \\ & \sin[\sigma_R])\dot{y}_{MT} + \cos[\sigma_E]\sin[\sigma_R]\dot{z}_{MT}\}\hat{y}_l + \\ & \{(\cos[\sigma_A]\cos[\sigma_R]\sin[\sigma_E] + \sin[\sigma_A]\sin[\sigma_R])\dot{x}_{MT} + (\cos[\sigma_R]\sin[\sigma_A]\sin[\sigma_E] - \cos[\sigma_A] \\ & \sin[\sigma_R])\dot{y}_{MT} + \cos[\sigma_E]\cos[\sigma_R]\dot{z}_{MT}\}\hat{z}_l\end{aligned} \quad [83]$$

The missile-target velocity vector relative to the earth-fixed system is also obtained by differentiating eq. 72 with respect to time t,

$$\begin{aligned}\vec{V}_{MT} = \dot{\vec{R}}_{MT} = \; & \{\cos[\sigma_A]\cos[\sigma_E]\dot{R}_{MT} - \cos[\sigma_E]\sin[\sigma_A]R_{MT}\dot{\sigma}_A - \cos[\sigma_A]\sin[\sigma_E]R_{MT}\dot{\sigma}_E\}\hat{x} + \\ & \{\cos[\sigma_E]\sin[\sigma_A]\dot{R}_{MT} + \cos[\sigma_A]\cos[\sigma_E]R_{MT}\dot{\sigma}_A - \sin[\sigma_A]\sin[\sigma_E]R_{MT}\dot{\sigma}_E\}\hat{y} + \\ & \{-\sin[\sigma_E]\dot{R}_{MT} - \cos[\sigma_E]R_{MT}\dot{\sigma}_E\}\hat{z}\end{aligned} \quad [84]$$

Equation 84 is converted to the LOS-fixed coordinate system using the Euler transformation based on the three Euler rotations σ_A, σ_E and σ_R, and the result is

$$\begin{aligned}\vec{V}_{MT} \;=\; & \dot{R}_{MT}\hat{x}_l \\ & +R_{MT}(\cos[\sigma_E]\cos[\sigma_R]\dot{\sigma}_A - \sin[\sigma_R]\dot{\sigma}_E)\hat{y}_l \\ & -R_{MT}(\cos[\sigma_E]\sin[\sigma_R]\dot{\sigma}_A + \cos[\sigma_R]\dot{\sigma}_E)\hat{z}_l\end{aligned} \quad [85]$$

where, as expected, the $\hat{x}_l$ component is $\dot{R}_{MT}$. Since eqs. 83 and 85 are equivalent, their $(\hat{x}_l, \hat{y}_l, \hat{z}_l)$ components are equated. The $\hat{x}_l$ term is denoted the closing velocity and is given by

$$\dot{R}_{MT} = \cos[\sigma_A]\cos[\sigma_E]\dot{x}_{MT} + \cos[\sigma_E]\sin[\sigma_A]\dot{y}_{MT} - \sin[\sigma_E]\dot{z}_{MT} \quad [86]$$

The $\hat{y}_l$ and $\hat{z}_l$ components are solved simultaneously for $\dot{\sigma}_A$ and $\dot{\sigma}_E$, namely,

$$\dot{\sigma}_A = -\frac{1}{R_{MT}}\{\sec[\sigma_E]\sin[\sigma_A]\dot{x}_{MT} - \cos[\sigma_A]\sec[\sigma_E]\dot{y}_{MT}\} \quad [87]$$

$$\dot{\sigma}_E = -\frac{1}{R_{MT}}(\cos[\sigma_A]\sin[\sigma_E]\dot{x}_{MT} + \sin[\sigma_A]\sin[\sigma_E]\dot{y}_{MT} + \cos[\sigma_E]\dot{z}_{MT}) \quad [88]$$

The simultaneous solution of the $\hat{y}_l$ and $\hat{z}_l$ components removes any dependence on σ_R.

The third component of the LOS angular rate vector, $\dot{\sigma}_R$, is obtained as follows. The angular rate vector of the LOS, $\vec{\omega}_l$, can be defined as

$$\vec{\omega}_l = \dot{\sigma}_R\hat{x}_{i2} + \dot{\sigma}_E\hat{x}_{i1} + \dot{\sigma}_A\hat{x} \quad [89]$$

It is converted to the LOS-fixed system using

$$\vec{\omega}_l = R_{\theta_R}[\sigma_R].R_{\theta_E}[\sigma_E].R_{\theta_A}[\sigma_A].\begin{bmatrix} 0 \\ 0 \\ \dot{\sigma}_A \end{bmatrix} + R_{\theta_R}[\sigma_R].R_{\theta_E}[\sigma_E].\begin{bmatrix} 0 \\ \dot{\sigma}_E \\ 0 \end{bmatrix} + R_{\theta_R}[\sigma_R].\begin{bmatrix} \dot{\sigma}_R \\ 0 \\ 0 \end{bmatrix} \quad [90]$$

or

$$\begin{aligned} \vec{\omega}_l = \; & -\sin[\sigma_E]\dot{\sigma}_A + \dot{\sigma}_R\hat{x}_l + \\ & \cos[\sigma_E]\sin[\sigma_R]\dot{\sigma}_A + \cos[\sigma_R]\dot{\sigma}_E\hat{y}_l + \\ & \cos[\sigma_E]\cos[\sigma_R]\dot{\sigma}_A - \sin[\sigma_R]\dot{\sigma}_E\hat{z}_l \end{aligned} \quad [91]$$

$\vec{\omega}_l$ is also expressed in terms of the non-Eulerian angular rate components relative to the earth-fixed reference system $(\omega_{xe}, \omega_{ye}, \omega_{ze})$, as

$$\vec{\omega}_l = \omega_{xe}\hat{x} + \omega_{ye}\hat{y} + \omega_{ze}\hat{z} = \frac{\vec{R}_{MT} \times \vec{V}_{MT}}{R_{MT}^2} \quad [92]$$

Converting eq. 92 to the LOS-fixed reference system,

$$\vec{\omega}_l = R_{\theta_R}[\sigma_R].R_{\theta_E}[\sigma_E].R_{\theta_A}[\sigma_A].\begin{bmatrix} \omega_{xe} \\ \omega_{ye} \\ \omega_{ze} \end{bmatrix} \quad [93]$$

Equating $\hat{x}_l$ components of eqs. 91 and 93 and substituting for $\dot{\sigma}_A$ from eq. 87,

$$\begin{aligned} \dot{\sigma}_R = \; & \cos[\sigma_A]\cos[\sigma_E]\omega_{xe} + \\ & \cos[\sigma_E]\sin[\sigma_A]\omega_{ye} - \sin[\sigma_E]\omega_{ze} - \frac{1}{R_{MT}}\sin[\sigma_E] \\ & (\sec[\sigma_E]\sin[\sigma_A]\dot{x}_{MT} - \cos[\sigma_A]\sec[\sigma_E]\dot{y}_{MT}) \end{aligned} \quad [94]$$

4.3 Tracking loop

The seeker non-Eulerian rotation rates in the seeker-fixed system, $\dot{\sigma}_r, \dot{\sigma}_{UD}$ and $\dot{\sigma}_{RL}$, are determined in terms of the Eulerian rates $\dot{\sigma}_A, \dot{\sigma}_E, \dot{\sigma}_R$, using the approach presented in Section 3.1. The Eulerian angular rotation vector $\vec{\omega}_l$ referenced to the earth-fixed system is given by

$$\vec{\omega}_l = \dot{\sigma}_R\hat{x}_{i2} + \dot{\sigma}_E\hat{y}_{i1} + \dot{\sigma}_A\hat{z} \quad [95]$$

The angular rotation vector expressed in the LOS-fixed system is

$$\vec{\omega}_l = \dot{\sigma}_r\hat{x}_l + \dot{\sigma}_{UD}\hat{y}_l + \dot{\sigma}_{RL}\hat{z}_l \quad [96]$$

Equation 95 is converted to the LOS-fixed coordinate system:

$$\vec{\omega}_l = R_{\theta_R}[\sigma_R].R_{\theta_E}[\sigma_E].R_{\theta_A}[\sigma_A].\begin{bmatrix} 0 \\ 0 \\ \dot{\sigma}_A \end{bmatrix} + R_{\theta_R}[\sigma_R].R_{\theta_E}[\sigma_E].\begin{bmatrix} 0 \\ \dot{\sigma}_E \\ 0 \end{bmatrix} + R_{\theta_R}[\sigma_R].\begin{bmatrix} \dot{\sigma}_R \\ 0 \\ 0 \end{bmatrix} \quad [97]$$

or

$$\vec{\omega}_l = \begin{array}{l}(-\sin[\sigma_E]\dot{\sigma}_A + \dot{\sigma}_R)\hat{x}_l + (\cos[\sigma_E]\sin[\sigma_R]\dot{\sigma}_A + \cos[\sigma_R]\dot{\sigma}_E)\hat{y}_l + \\ (\cos[\sigma_E]\cos[\sigma_R]\dot{\sigma}_A - \sin[\sigma_R]\dot{\sigma}_E)\hat{z}_l\end{array} \quad [98]$$

Equating the $\hat{y}_l, \hat{z}_l$ components of eqs. 96 and 98,

$$\dot{\sigma}_{RL} = \cos[\sigma_E]\cos[\sigma_R]\dot{\sigma}_A - \sin[\sigma_R]\dot{\sigma}_E \quad [99]$$

$$\dot{\sigma}_{UD} = \cos[\sigma_E]\sin[\sigma_R]\dot{\sigma}_A + \cos[\sigma_R]\dot{\sigma}_E \quad [100]$$

$$\dot{\sigma}_r = -\sin[\sigma_E]\dot{\sigma}_A + \dot{\sigma}_R \quad [101]$$

The above relations describe the LOS angular rates $\dot{\sigma}_r, \dot{\sigma}_{RL}, \dot{\sigma}_{UD}$ referenced to the LOS-fixed system in terms of the LOS Euler angular rates $\dot{\sigma}_A, \dot{\sigma}_E, \dot{\sigma}_R$ referenced to the earth-fixed system.

The angular separation between the LOS vector and the gyro spin axis (seeker boresight) is denoted $\vec{\epsilon}$ and its components are ϵ_{RL} and ϵ_{UD}. The components represent Euler rotations in sequence from the seeker-fixed system to the LOS-fixed one, namely,

$$\vec{\epsilon} = \epsilon_{UD}\hat{y}_{i1} + \epsilon_{RL}\hat{z}_s \quad [102]$$

The total tracking error, ϵ, is defined as

$$\epsilon = \sqrt{\epsilon_{RL}^2 + \epsilon_{UD}^2} \quad [103]$$

The tracking rate $\dot{\phi}$ is proportional to the total tracking error. As a first approximation, the tracking rate $\dot{\phi}$ can be modeled in a generic way using

$$\dot{\phi} = K_p \epsilon H(s) \quad [104]$$

where $H(s)$ is the seeker tracking loop transfer function and K_p is a constant gain.

The direction of the tracking rate vector, $\dot{\vec{\phi}}$, is given by $\hat{x}_s \times \hat{x}_l$, that is, along the vector perpendicular to the plane formed by $\hat{x}_s$ and $\hat{x}_l$. The unit vector $\hat{x}_p$ is defined to be in this direction, namely,

$$\hat{x}_p = \frac{\hat{x}_s \times \hat{x}_l}{|\hat{x}_s \times \hat{x}_l|} \quad [105]$$

$\dot{\vec{\phi}}$ is hence

$$\dot{\vec{\phi}} = K_p \epsilon H(s)\hat{x}_p \quad [106]$$

The term $\hat{x}_s \times \hat{x}_l$ is obtained from the seeker-fixed to the LOS-fixed Euler transformation $(\epsilon_{UD}, \epsilon_{RL})$. The $\hat{x}_l$ component of the transformation $R_{\theta_E}[\epsilon_{UD}].R_{\theta_A}[\epsilon_{RL}]$ is

$$\hat{x}_l = \cos[\epsilon_{RL}]\cos[\epsilon_{UD}]\hat{x}_s + \cos[\epsilon_{UD}]\sin[\epsilon_{RL}]\hat{y}_s - \sin[\epsilon_{UD}]\hat{z}_s \quad [107]$$

Accordingly,

$$\hat{x}_s \times \hat{x}_l = \sin[\epsilon_{UD}]\hat{y}_s + \cos[\epsilon_{UD}]\sin[\epsilon_{RL}]\hat{z}_s \quad [108]$$

and

$$|\hat{x}_s \times \hat{x}_l|^2 = \cos[\epsilon_{UD}]^2 \sin[\epsilon_{RL}]^2 + \sin[\epsilon_{UD}]^2 \quad [109]$$

For small angles, $\cos[\epsilon] \approx 1$, and $\sin[\epsilon] \approx \epsilon$,

$$\hat{x}_s \times \hat{x}_l \approx \epsilon_{UD}\hat{y}_s + \epsilon_{RL}\hat{z}_s \quad [110]$$

and, using eq. 103,

$$|\hat{x}_s \times \hat{x}_l| \approx \epsilon \quad [111]$$

Accordingly, eq. 105 for $\hat{x}_p$ reduces to

$$\hat{x}_p = \frac{\epsilon_{UD}}{\epsilon}\hat{y}_s + \frac{\epsilon_{RL}}{\epsilon}\hat{z}_s \quad [112]$$

Finally, substituting eq. 112 in eq. 106, $\dot{\vec{\phi}}$ referenced to the seeker-fixed system is

$$\dot{\vec{\phi}} = K_p H(s)\left[\epsilon_{UD}\hat{y}_s + \epsilon_{RL}\hat{z}_s\right] \quad [113]$$

The components of $\dot{\vec{\phi}}$, denoted $\dot{\phi}_{UD}$ and $\dot{\phi}_{RL}$, are

$$\dot{\vec{\phi}} = \dot{\phi}_{UD}\hat{y}_s + \dot{\phi}_{RL}\hat{z}_s \quad [114]$$

In comparison, the seeker non-Eulerian angular rate referenced to the seeker-fixed system is

$$\vec{\omega}_s = \dot{\phi}_r\hat{x}_s + \dot{\phi}_{UD}\hat{y}_s + \dot{\phi}_{RL}\hat{z}_s \quad [115]$$

The rates $\dot{\epsilon}_{RL}$ and $\dot{\epsilon}_{UD}$ are derived using the procedure outlined in Section 3.3. The LOS angular rate vector $\vec{\omega}_l$ is defined in reference to the seeker-fixed system as

$$\vec{\omega}_l = \vec{\omega}_s + \vec{\omega}_{l/s} \quad [116]$$

The LOS non-Eulerian angular rate referenced to the LOS-fixed system is

$$\vec{\omega}_l = \dot{\sigma}_r\hat{x}_l + \dot{\sigma}_{UD}\hat{y}_l + \dot{\sigma}_{RL}\hat{z}_l \quad [117]$$

The Eulerian angular rate vector of the LOS referenced to the seeker-fixed system is denoted as $\vec{\omega}_{l/s}$ and is defined as

$$\vec{\omega}_{l/s} = \dot{\epsilon}_{UD}\hat{y}_{i1} + \dot{\epsilon}_{RL}\hat{z}_s \quad [118]$$

Equations 115, 117 and 118 are converted to a single coordinate system, namely, the LOS-fixed system. $\vec{\omega}_s$ is converted using the transformation $R_{\theta_E}[\epsilon_{UD}].R_{\theta_A}[\epsilon_{RL}]$, namely,

$$\begin{aligned}\vec{\omega}_s = &\{\cos[\epsilon_{RL}]\cos[\epsilon_{UD}]\dot{\phi}_r - \sin[\epsilon_{UD}]\dot{\phi}_{RL} + \cos[\epsilon_{UD}]\sin[\epsilon_{RL}]\dot{\phi}_{UD}\}\hat{x}_l + \\ &\{-\sin[\epsilon_{RL}]\dot{\phi}_r + \cos[\epsilon_{RL}]\dot{\phi}_{UD}\}\hat{y}_l + \\ &\{\cos[\epsilon_{RL}]\sin[\epsilon_{UD}]\dot{\phi}_r + \cos[\epsilon_{UD}]\dot{\phi}_{RL} + \sin[\epsilon_{RL}]\sin[\epsilon_{UD}]\dot{\phi}_{UD}\}\hat{z}_l\end{aligned} \qquad [119]$$

The LOS angular rate vector $\vec{\omega}_l$ is already in the LF system. $\vec{\omega}_l/s$ is obtained from the transformation

$$\vec{\omega}_{l/s} = R_{\theta_E}[\epsilon_{UD}].R_{\theta_A}[\epsilon_{RL}].\begin{bmatrix} 0 \\ 0 \\ \dot{\epsilon}_{RL} \end{bmatrix} + R_{\theta_E}[\epsilon_{UD}].\begin{bmatrix} 0 \\ \dot{\epsilon}_{UD} \\ 0 \end{bmatrix} \qquad [120]$$

or

$$\vec{\omega}_{l/s} = -\sin[\epsilon_{UD}]\dot{\epsilon}_{RL}\hat{x}_l + \dot{\epsilon}_{UD}\hat{y}_l + \cos[\epsilon_{UD}]\dot{\epsilon}_{RL}\hat{z}_l \qquad [121]$$

Equations 117, 119 and 121 are substituted in eq. 116. Equating the $\hat{y}_l$ components and, given that the maximum angular separation ϵ is usually less than 10 degrees, the resulting expression for $\dot{\epsilon}_{UD}$ is

$$\dot{\epsilon}_{UD} = \dot{\sigma}_{UD} + \epsilon_{RL}\dot{\phi}_r - \dot{\phi}_{UD} \qquad [122]$$

Similarly, equating the $\hat{z}_l$ components,

$$\dot{\epsilon}_{RL} = \dot{\sigma}_{RL} - \epsilon_{UD}\dot{\phi}_r - \dot{\phi}_{RL} \qquad [123]$$

Equations 122 and 123 are integrated to give ϵ_{RL} and ϵ_{UD} as a function of time. The total error ϵ is computed from eq. 103 and substituted in eq. 113 to give the total tracking rate vector $\dot{\vec{\phi}}$ (specifically, $\dot{\phi}_{UD}, \dot{\phi}_{RL}$).

The total tracking rate will usually be limited, and the limited value is denoted as $\dot{\phi}_l$. The components of $\dot{\vec{\phi}}_l$ are

$$\dot{\vec{\phi}}_l = \dot{\phi}_l\hat{x}_p \qquad [124]$$

or, using eq. 112,

$$\dot{\vec{\phi}}_l = \frac{\epsilon_{UD}}{\epsilon}\dot{\phi}_l\hat{y}_s + \frac{\epsilon_{RL}}{\epsilon}\dot{\phi}_l\hat{z}_s \qquad [125]$$

5.0 SIMULATION CONTROL

5.1 Initial conditions

The initial conditions are obtained from the following specified parameters:

- V_M, V_T
- σ_A, σ_E
- θ_A , θ_E , θ_R
- Γ_A, Γ_E ($\Gamma_R = 0$)
- R_{MT}
- h_T (target altitude)
- γ_A, γ_E ($\gamma_R = 0$)

The initial components of the missile incidence, α_A, α_E are obtained from eqs. 45 and 46. The relations for the initial values of $\alpha_{UD}, \alpha_{RL}, \lambda_{UD}, \lambda_{RL}, \phi_A, \phi_E$ and ϕ_R are summarized. The angles α_{UD} and α_{RL} are computed from eqs. 43 and 44,

$$\alpha_{UD} = \cos[\gamma_E]\sin[\theta_R]\alpha_A + \cos[\theta_R]\alpha_E \qquad [126]$$

and

$$\alpha_{RL} = \cos[\gamma_E]\cos[\theta_R]\alpha_A - \sin[\theta_R]\alpha_E \qquad [127]$$

Seeker lock-on is assumed at missile launch. The gyro axis $\hat{x}_s$ is hence aligned with the LOS vector and thus corresponds to $\hat{x}_l$. $\hat{x}_l$ is given by the following rotation from the earth-fixed system,

$$\begin{bmatrix} \hat{x}_l \\ \hat{y}_l \\ \hat{z}_l \end{bmatrix} = R_{\theta_R}[\sigma_R].R_{\theta_E}[\sigma_E].R_{\theta_A}[\sigma_A] \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} \qquad [128]$$

The transformation from the earth-fixed system to the body-fixed one is given by

$$\begin{bmatrix} \hat{x}_b \\ \hat{y}_b \\ \hat{z}_b \end{bmatrix} = R_{\theta_R}[\theta_R].R_{\theta_E}[\theta_E].R_{\theta_A}[\theta_A] \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} \qquad [129]$$

The unit vector $\hat{x}_b$ is therefore transformed to the LOS-fixed system using

$$\begin{bmatrix} \hat{x}_l|_{\hat{x}_b} \\ \hat{y}_l|_{\hat{x}_b} \\ \hat{z}_l|_{\hat{x}_b} \end{bmatrix} = R_{\theta_R}[\sigma_R].R_{\theta_E}[\sigma_E].R_{\theta_A}[\sigma_A].R_{-\theta_A}[\theta_A].R_{-\theta_E}[\theta_E].R_{-\theta_R}[\theta_R].\begin{bmatrix} \hat{x}_b \\ 0 \\ 0 \end{bmatrix} \quad [130]$$

The contribution of $\hat{x}_b$ to $\hat{x}_l$, denoted $\hat{x}_l|_{\hat{x}_b}$, is, from eq. 130,

$$\begin{aligned}\hat{x}_l|_{\hat{x}_b} = \; & (\cos[\theta_E](\cos[\theta_A]\cos[\sigma_A]\cos[\sigma_E] + \cos[\sigma_E]\sin[\theta_A]\sin[\sigma_A]) + \\ & \sin[\theta_E]\sin[\sigma_E])\hat{x}_b\end{aligned} \quad [131]$$

Similarly, the contribution of $\hat{y}_b$ to $\hat{x}_l$ (eq. 130), denoted as $\hat{x}_l|_{\hat{y}_b}$, is

$$\begin{aligned}\hat{x}_l|_{\hat{y}_b} = \; & (\cos[\theta_R](-\cos[\sigma_A]\cos[\sigma_E]\sin[\theta_A] + \cos[\theta_A]\cos[\sigma_E]\sin[\sigma_A]) + \sin[\theta_R] \\ & (\sin[\theta_E](\cos[\theta_A]\cos[\sigma_A]\cos[\sigma_E] + \cos[\sigma_E]\sin[\theta_A]\sin[\sigma_A]) - \cos[\theta_E]\sin[\sigma_E]))\hat{y}_b\end{aligned} \quad [132]$$

and the contribution of $\hat{z}_b$ to $\hat{x}_l$ (eq. 130), denoted as $\hat{x}_l|_{\hat{z}_b}$, is

$$\begin{aligned}\hat{x}_l|_{\hat{z}_b} = \; & (-\sin[\theta_R](-\cos[\sigma_A]\cos[\sigma_E]\sin[\theta_A] + \cos[\theta_A]\cos[\sigma_E]\sin[\sigma_A]) + \cos[\theta_R] \\ & (\sin[\theta_E](\cos[\theta_A]\cos[\sigma_A]\cos[\sigma_E] + \cos[\sigma_E]\sin[\theta_A]\sin[\sigma_A]) - \cos[\theta_E]\sin[\sigma_E]))\hat{z}_b\end{aligned} \quad [133]$$

The contributions of $\hat{x}_b, \hat{y}_b, \hat{z}_b$ to $\hat{x}_l$ are also obtained from the missile body to seeker transformation. The transformation of $\hat{x}_b$ to the seeker-fixed system is performed using the transformation

$$\begin{bmatrix} \hat{x}_s|_{\hat{x}_b} \\ \hat{y}_s|_{\hat{x}_b} \\ \hat{z}_s|_{\hat{x}_b} \end{bmatrix} = R_{\theta_E}[\lambda_{UD}].R_{\theta_A}[\lambda_{RL}].\begin{bmatrix} \hat{x}_b \\ 0 \\ 0 \end{bmatrix} \quad [134]$$

This is also the contribution to the LOS-fixed system since the two systems are initially co-located:

$$\begin{bmatrix} \hat{x}_l|_{\hat{x}_b} \\ \hat{y}_l|_{\hat{x}_b} \\ \hat{z}_l|_{\hat{x}_b} \end{bmatrix} = \begin{bmatrix} \hat{x}_s|_{\hat{x}_b} \\ \hat{y}_s|_{\hat{x}_b} \\ \hat{z}_s|_{\hat{x}_b} \end{bmatrix} \quad [135]$$

Accordingly, the contribution of $\hat{x}_b$ to $\hat{x}_l$ (eq. 134) is

$$\hat{x}_l|_{\hat{x}_b} = \cos[\lambda_{RL}]\cos[\lambda_{UD}]\hat{x}_b \quad [136]$$

Similarly, the contribution of $\hat{y}_b$ to $\hat{x}_l$ (eq. 134) is

$$\hat{x}_l|_{\hat{y}_b} = \cos[\lambda_{UD}]\sin[\lambda_{RL}]\hat{y}_b \quad [137]$$

and the contribution of $\hat{z}_b$ to $\hat{x}_l$ (eq. 134) is

$$\hat{x}_l|_{\hat{z}_b} = -\sin[\lambda_{UD}]\hat{z}_b \quad [138]$$

From the eqs. 136, 137 and 138, $-\hat{x}_l|_{\hat{z}_b}$ equals $\sin[\lambda_{UD}]$ and the ratio of $\hat{x}_l|_{\hat{y}_b}$ to $\hat{x}_l|_{\hat{x}_b}$ equals $\tan[\lambda_{RL}]$.

Since eqs. 131, 132 and 133 correspond to eqs. 136, 137 and 138 respectively, $\sin[\lambda_{UD}]$ is given by

$$\begin{aligned}\sin[\lambda_{UD}] = \; & \sin[\theta_R](-\cos[\sigma_A]\cos[\sigma_E]\sin[\theta_A] + \cos[\theta_A]\cos[\sigma_E]\sin[\sigma_A]) - \cos[\theta_R] \\ & (\sin[\theta_E](\cos[\theta_A]\cos[\sigma_A]\cos[\sigma_E] + \cos[\sigma_E]\sin[\theta_A]\sin[\sigma_A]) - \cos[\theta_E]\sin[\sigma_E])\end{aligned} \quad [139]$$

and $\tan[\lambda_{RL}]$, by

$$\begin{aligned}\tan[\lambda_{RL}] = \; & (\cos[\theta_R](-\cos[\sigma_A]\cos[\sigma_E]\sin[\theta_A] + \cos[\theta_A]\cos[\sigma_E]\sin[\sigma_A]) + \sin[\theta_R] \\ & (\sin[\theta_E](\cos[\theta_A]\cos[\sigma_A]\cos[\sigma_E] + \cos[\sigma_E]\sin[\theta_A]\sin[\sigma_A]) - \cos[\theta_E]\sin[\sigma_E]))/ \\ & (\cos[\theta_E](\cos[\theta_A]\cos[\sigma_A]\cos[\sigma_E] + \cos[\sigma_E]\sin[\theta_A]\sin[\sigma_A]) + \sin[\theta_E]\sin[\sigma_E])\end{aligned} \quad [140]$$

The orientation of the seeker coordinate system relative to the earth-fixed system is given by the Euler angles ϕ_A, ϕ_E, ϕ_R. Since the seeker-fixed and LOS-fixed systems are co-located at launch, the seeker boresight error angles, $\epsilon_{UD}, \epsilon_{RL}$, are null and the seeker boresight Euler angles ϕ_A and ϕ_E are

$$\phi_A = \sigma_A \quad [141]$$

and

$$\phi_E = \sigma_E \quad [142]$$

relative to the earth-fixed system.

The initial roll angle ϕ_R is obtained next using two expressions, one derived from the LOS-fixed system and a second one from the seeker-fixed system. Seeker lock-on at launch implies that ϕ_R, the Euler roll angle of the seeker-fixed system, equals σ_R, the Euler roll angle of the LOS-fixed system. First, the intermediate II coordinate system of the Euler transformation from the earth-fixed system to the LOS-fixed one is considered. The intermediate II system ($\hat{x}_{i2}$, $\hat{y}_{i2}$, $\hat{z}_{i2}$) is obtained by rotating the earth-fixed system $(\hat{x}, \hat{y}, \hat{z})$ through the Euler rotations σ_A and σ_E in sequence. The $\hat{y}_{i2}$ term is

$$\hat{y}_{i2} = -\sin[\sigma_A]\hat{x} + \cos[\sigma_A]\hat{y} \quad [143]$$

The unit vector $\hat{y}_{i2}$ is now expressed in the body-fixed reference system by first rotating back to the earth-fixed system and forward to the body-fixed system according to

$$\begin{bmatrix} \hat{x}_b|_{\hat{y}_{i2}} \\ \hat{y}_b|_{\hat{y}_{i2}} \\ \hat{z}_b|_{\hat{y}_{i2}} \end{bmatrix} = R_{\theta_R}[\theta_R].R_{\theta_E}[\theta_E].R_{\theta_A}[\theta_A].R_{-\theta_A}[\sigma_A].R_{-\theta_E}[\sigma_E].\begin{bmatrix} 0 \\ \hat{y}_{i2} \\ 0 \end{bmatrix} \quad [144]$$

or

$$\begin{bmatrix} \hat{x}_b|_{\hat{y}_{i2}} \\ \hat{y}_b|_{\hat{y}_{i2}} \\ \hat{z}_b|_{\hat{y}_{i2}} \end{bmatrix} = \begin{bmatrix} \{\cos[\theta_E]\cos[\sigma_A]\sin[\theta_A] - \cos[\theta_A]\cos[\theta_E]\sin[\sigma_A]\}, \\ \{\cos[\sigma_A](\cos[\theta_A]\cos[\theta_R] + \sin[\theta_A]\sin[\theta_E]\sin[\theta_R]) - \\ (-\cos[\theta_R]\sin[\theta_A] + \cos[\theta_A]\sin[\theta_E]\sin[\theta_R])\sin[\sigma_A]\}, \\ \{\cos[\sigma_A](\cos[\theta_R]\sin[\theta_A]\sin[\theta_E] - \cos[\theta_A]\sin[\theta_R]) - \\ (\cos[\theta_A]\cos[\theta_R]\sin[\theta_E] + \sin[\theta_A]\sin[\theta_R])\sin[\sigma_A]\} \end{bmatrix} \hat{y}_{i2} \qquad [145]$$

from which the first expression for $\hat{y}_{i2}.\hat{z}_b$ is obtained.

A second expression for $\hat{y}_{i2}.\hat{z}_b$ is obtained by considering the seeker-fixed system. The transformation from the seeker-fixed system to the body-fixed one is

$$\begin{bmatrix} \hat{x}_b \\ \hat{y}_b \\ \hat{z}_b \end{bmatrix} = R_{-\theta_A}[\lambda_{RL}].R_{-\theta_E}[\lambda_{UD}]. \begin{bmatrix} \hat{x}_s \\ \hat{y}_s \\ \hat{z}_s \end{bmatrix} \qquad [146]$$

where the $\hat{z}_b$ term is

$$\hat{z}_b = -\sin[\lambda_{UD}]\hat{x}_s + \cos[\lambda_{UD}]\hat{z}_s \qquad [147]$$

The transformation from the seeker-fixed to the intermediate II system is only $R_{-\theta_R}[\phi_R]$, or equivalently, $R_{-\theta_R}[\sigma_R]$, as noted previously. The $\hat{y}_{i2}$ component is hence

$$\hat{y}_{i2} = \cos[\phi_R]\hat{y}_s - \sin[\phi_R]\hat{z}_s \qquad [148]$$

The second expression for $\hat{y}_{i2}.\hat{z}_b$ is obtained from eqs. 147 and 148, namely,

$$\hat{y}_{i2}.\hat{z}_b = -\cos[\lambda_{UD}]\sin[\phi_R] \qquad [149]$$

Combining with eq. 145 and solving for $\sin[\phi_R]$,

$$\begin{aligned} \sin[\phi_R] = \; & -\sec[\lambda_{UD}](\cos[\sigma_A](\cos[\theta_R]\sin[\theta_A]\sin[\theta_E] - \cos[\theta_A]\sin[\theta_R]) - \\ & (\cos[\theta_A]\cos[\theta_R]\sin[\theta_E] + \sin[\theta_A]\sin[\theta_R])\sin[\sigma_A]) \end{aligned} \qquad [150]$$

5.2 Miss distance termination criterion

The estimated missile-target miss distance is computed based on the assumption that all missile and target accelerations cease from the time of estimation to the point of closest approach. The relative velocity vector is then projected forward to define the point of closest approach between the missile and the target. This separation is the projected miss distance.

From geometrical considerations, the miss distance d_{miss} is given by

$$d_{miss} = R_{MT}\frac{\left|R_{MT}\dot{\hat{x}}_l\right|}{\left|\vec{V}_{MT}\right|} \qquad [151]$$

The expression for $R_{MT}\dot{\hat{x}}_l$ is obtained from the range vector in the LOS-fixed coordinate system. Since

$$\vec{R}_{MT} = R_{MT}\hat{x}_l \qquad [152]$$

then,

$$R_{MT}\dot{\hat{x}}_l = \vec{V}_{MT} - \dot{R}_{MT}\hat{x}_l \qquad [153]$$

Substituting eq. 153 for $R_{MT}\dot{\hat{x}}_l$ and eq. 85 for $\vec{V}_{MT}$ into eq. 151, the resulting expression for the miss distance is

$$d_{miss} = \frac{R_{MT}^2\sqrt{\cos^2[\sigma_E]\dot{\sigma}_A^2 + \dot{\sigma}_E^2}}{\sqrt{\dot{R}_{MT}^2 + R_{MT}^2(\cos^2[\sigma_E]\dot{\sigma}_A^2 + \dot{\sigma}_E^2)}} \qquad [154]$$

6.0 CONCLUDING REMARKS

DREV is currently supporting the development of high-fidelity simulation models of air-to-air tactical missiles for use in weapon system effectiveness assessment. For efficiency, the development of this missile simulation capability takes advantage of already existing model components when available, one example being the modeling of the missile dynamics. The available documented model of the missile dynamics uses a non-standard axes system where x is forward, z is lateral and y is upward. This model was modified to adopt the standard NED axes system (x forward, y lateral, z downward) to ensure compatibility with existing simulation models and, in particular, with the DREV library of missile components. The mathematical relations of the missile dynamics model based on the NED axes system have been developed and presented. This description covered the Euler transformations between the six coordinate systems required for the missile-target engagement simulation, the dynamics of the rigid airframe and of the seeker (the tracking loop), and the computation of the initial conditions and of the miss distance. The next step in the development of the required simulation capability is the software implementation of this missile dynamics model.

7.0 REFERENCES

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The Defence Research Establishment Valcartier currently supports the development of a generic six-degree-of-freedom simulation model of a tactical air-to-air missile.
For efficiency, the development of this missile simulation takes advantage of already existing model components when available, such as in the case of the missile dynamics.
The documented 6DOF missile dynamics model currently available uses a non-standard axis system, where the z axis is lateral and the y axis is downward, to describe the missile body, seeker and missile-target line-of-sight vector.
However, the most common axis system in current simulation models and in the DREV library of weapon system model components is the NED system (North-East-Down), which uses a forward x axis, a downward z axis and a lateral y axis.
In order to maintain compatibility with the existing library of components, this missile dynamics model has been modified to use the NED standard system.
The mathematical relations of the resulting model are presented.

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